Minimal surfaces of the form $f(x) + g(y) + h(z) = 0$

Consider the non-linear partial differential equation

$$z_{xx}(1 + z_y^2) + z_{yy}(1 + z_x^2) = 2z_x z_y z_{xy}, \tag{1}$$

whose solutions $z(x, y)$ correspond to 'minimal' surfaces in $\mathbb{R}^3$ locally described as graphs, i.e. giving the height z over a point (x, y). Inserting the Ansats

$$z(x, y) = \zeta(f(x) + g(y)) \tag{2}$$

into (1) gives a differential equation (*) in which the first and second derivatives of the 3 functions f, g and ζ appear (each being a function of one variable). Using the notation $-h$ for the inverse of ζ, i.e. with $\zeta(-h(z)) = z$, show that (*) becomes

$$f''(x)(g'(y)^2 + h'(z)^2) + g''(h'^2 + f'^2) + h''(f'^2 + g'^2) = 0, \tag{3}$$

-which does not have to hold for *all* $x, y, z \in \mathbb{R}$, but only for those satisfying

$$f(x) + g(y) + h(z) = 0. \tag{4}$$

Find solutions (the more the better) of (3)/(4) assuming that, with some constants $\gamma, a_i, b_i, c_i (i = 1, 2, 3)$

$$f_i'^2(x_i) = a_i + b_i e^{\gamma f_i} + c_i e^{-\gamma f_i}, \tag{5}$$

-having switched here to the notation $x_1 = x, x_2 = y, x_3 = z, f_1 = f, f_2 = g, f_3 = h$.What conditions do the $a_i, b_i, c_i, i = 1, 2, 3$, have to satisfy so that the Ansats (5) solves (3), resp. (2) solves (1). Show that the 4 surfaces given by

$$S_1 : \quad y \cos z = x \sin z \tag{6}$$

$$S_2 : \quad x^2 + y^2 = (\cosh z)^2 \tag{7}$$

$$S_3 : \quad \sin z = \sinh x \sinh y \tag{8}$$

$$S_4 : \quad e^z \cos x = \cos y \tag{9}$$

are of the form (4), and (except for S_2 for which (3) can be verified directly) for appropriately chosen f_i and constants a_i, b_i, c_i, γ satisfy (5), resp. (3). Try to draw (at least sketch) the 4 surfaces, resp. describe some of their properties in words.